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SHELLINGS OF SIMPLICIAL BALLS AND P.L. MANIFOLDS WITH BOUNDARY

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Shellability of simplicial complexes has been a powerful concept in polyhydral theory, in p.l. topology and recently in connection with Cohen–Macaulay rings. It is known that all 2-spheres and all boundary complexes of convex polytopes are shellable. The analogous theorem fails for general simplicial balls and spheres.

In this paper we study transformations of simplicial p.l. manifolds by elementary boundary operations (shellings and inverse shellings) and bistellar operations (the inner equivalent to shellings). It is shown that a simplicial p.l. manifold $\mathcal{M}$ can be transformed in any other simplicial p.l. manifold $\mathcal{M}'$ homeomorphic to $\mathcal{M}$ using these elementary operations. In the case of balls only elementary boundary operations are needed.

1. Introduction

Shellability of simplicial complexes has proved to be a useful concept in polyhedral theory, in piecewise linear topology and recently in connection with Cohen–Macaulay rings [2, 4, 19, 3, 15]. For a survey the reader can consult [6].

It has turned out that there exist non-shellable simplicial p.l. balls and spheres. So there is some need for new construction methods which might yield generalizations of result about shellable complexes. In fact some of these results have been generalized but only with the heavy machinery of commutative algebra [28, 29].

In this paper we shall show that every simplicial p.l. ball is obtainable from the simplex by successive "shelling up and down". Similar transformations are presented for homeomorphic p.l. manifolds with boundary.

2. Basic concepts

Let P be a (convex) polytope. The boundary complex of P is denoted by $\mathcal{B}(P)$ and $\mathcal{F}(P) := \mathcal{B}(P) \cup \{P\}$. For a single point p we write $\mathcal{F}(\{p\}) =: \bar{p}$. For more information about polytopes the reader is referred to [12]. In the sequal T^d always denotes a d-dimensional simplex.

A finite simplicial complex $\mathscr{C}$ is defined in the usual way. The members of $\mathscr{C}$ are the *faces* of $\mathscr{C}$ and $\dim A$ denotes the dimension of a face A of $\mathscr{C}$. $\mathscr{C}$ is a simplicial n-complex if n is the maximum dimension of its faces. We use the following notations:

$\mathrm{st}(A, \mathscr{C}) := \{B \in \mathscr{C} : A \subseteq B\}$ "*(open) star*"
$\mathrm{clst}(A; \mathscr{C}) := \bigcup \{\mathscr{F}(B) : B \in \mathrm{st}(A; \mathscr{C})\}$ "*(closed) star*"
$\mathrm{ast}(A; \mathscr{C}) := \{B \in \mathscr{C} : B \cap A = \emptyset\}$ "*antistar*"
$\mathrm{link}(A; \mathscr{C}) := \mathrm{ast}(A; \mathscr{C}) \cap \mathrm{clst}(A; \mathscr{C})$
$\Delta_k(\mathscr{C}) := \{A \in \mathscr{C}; \dim A = k\}$
$\mathrm{skel}_k(\mathscr{C}) := \{A \in \mathscr{C} : \mathrm{din}\, A \leqslant k\}$ "*k-skeleton*"
$\mathrm{vert}(\mathscr{C}) := \Delta_1(\mathscr{C})$ "*vertices*"
$|\mathscr{C}| := \bigcup \mathscr{C}$ "*underlying topological space*"

The maximal faces of $\mathscr{C}$ are the *facets* of $\mathscr{C}$. A *missing face* of $\mathscr{C}$ is a simplex $D \notin \mathscr{C}$ with $\mathscr{B}(D) \subseteq \mathscr{C}$. A simplicial n-complex $\mathscr{M}$ is called a simplicial n-ball, sphere or manifold if $|\mathscr{M}|$ is a ball, a sphere or a manifold, respectively (all balls, spheres, manifolds and homeomorphisms to be considered are piecewise linear). $\mathrm{Bd}(\mathscr{M})$ denotes the *boundary complex* of a simplicial n-manifold $\mathscr{M}$. This is the subcomplex of $\mathscr{M}$ which has as facets those $(n-1)$-faces of $\mathscr{M}$ which are contained in only one facet of $\mathscr{M}$. The set of the *inner faces* of $\mathscr{M}$ is denoted by $\mathrm{Int}(\mathscr{M}) := \mathscr{M} \setminus \mathrm{Bd}(\mathscr{M})$. We use "$\cong$" both for homeomorphic polyhedrons and for isomorphic complexes. But, because additional isomorphisms are always allowed (and often necessary) we shall mostley write "$=$" instead of "$\cong$".

The *join* of simplicial complexes $\mathscr{C}$, $\mathscr{C}'$ is defined by $\mathscr{C} \cdot \mathscr{C}' := \{A \cdot A' : A \in \mathscr{C}, A' \in \mathscr{C}'\}$ where $A \cdot A' := \mathrm{conv}(A \cup A')$ is the convex hull. Here we always assume that $|\mathscr{C}|$, $|\mathscr{C}'|$ are joinable (see [11, 14]). This is, for instance, the case if $|\mathscr{C}|$, $|\mathscr{C}'|$ are embedded into disjoint affine subspaces containing no parallel lines. The joint of subsets of joinable complexes is defined in the obvious way.

Definition 1. (1) Let $\mathscr{M}$ be a simplicial n-manifold, and let $F = A \cdot B$ be a facet of $\mathscr{M}$ such that $A \in \mathrm{Int}(\mathscr{M})$, $\mathscr{B}(A) \cdot B \subseteq \mathrm{Bd}(\mathscr{M})$ and $\dim A$, $\dim B \geqslant 0$. Then we call

$$\mathscr{M}' := \rho_{-F}\mathscr{M} := \mathscr{M} \setminus \mathscr{F}(A) \cdot B$$

an (*elementary*) *shelling* of $\mathscr{M}$.

The inverse operation is denoted by $\rho_{+F}\mathscr{M} := \rho_{-F}^{-1}\mathscr{M}$

(2) $\mathscr{M} \xrightarrow{\mathrm{sh}} \mathscr{M}' :\Leftrightarrow \mathscr{M}' = \rho_r \ldots \rho_1\mathscr{M}$ where the ρ_k-s are elementary shellings.

$\mathscr{M} \approx_{\mathrm{sh}\pm} \mathscr{M}' :\Leftrightarrow \mathscr{M}' = \rho_r \ldots \rho_1\mathscr{M}$, where ρ_k is an elementary shelling or an inverse elementary shelling for $k = 1, \ldots, r$.

(3) $\mathscr{K}$ is a *shellable simplicial n-ball* $:\Leftrightarrow \mathscr{K} \xrightarrow{\mathrm{sh}} \mathscr{F}(T^n)$.

A simplicial n-sphere $\mathscr{S}$ is called shellable iff there exists a facet F of $\mathscr{S}$ such that $\mathscr{S} \setminus \{F\}$ is a shellable n-ball.

Remarks and additional notations. (1) It can happen that there exists a face $A \in \mathrm{Int}(\mathcal{M})$ and different faces B_1, B_2 such that $\mathcal{B}(A) \cdot B_1$, $\mathcal{B}(A) \cdot B_2 \subseteq \mathrm{Bd}(\mathcal{M})$ and $A \cdot B_1$, $A \cdot B_2$ are both facets of $\mathcal{M}$. But for every $B \in \mathrm{Bd}(\mathcal{M})$ there exists at most one $A \in \mathrm{Int}(\mathcal{M})$ such that $A \cdot B$ is a facet of $\mathcal{M}$ with $\mathcal{B}(A) \cdot B \subseteq \mathrm{Bd}(\mathcal{M})$. So ρ_{-F} is uniquely determined by B and we write $\rho_{-F} =: \rho_{-B}$. Analogously we write ρ_{+A} for an inverse elementary shelling. This implies that $A \in \mathrm{Bd}(\mathcal{M})$ and $\mathrm{link}(A; \mathrm{Bd}(\mathcal{M})) = \mathcal{B}(B)$ for a missing face B of $\mathcal{M}$.

(2) $\mathcal{M} \xrightarrow{\mathrm{sh}} \mathcal{M}'$ as well as $\mathcal{M}' \approx_{\mathrm{sh}\pm} \mathcal{M}'$ imply $|\mathcal{M}| \cong |\mathcal{M}'|$.
Obviously "$\cong_{\mathrm{sh}\pm}$" is an equivalence relation.

We shall mention here only some of these known important results about shellings, which are relevant in the context of the present paper.

(1.1) Boundary complexes of (simplicial) polytopes are shellable (Bruggesser/Mani [5]).

(1.2) There exist non-shellable simplicial balls (Rudin [26], Grünbaum [13]).

(1.3) There exist non-shellable triangulated topological 5-spheres (Edwards [7]).

(1.4) Every simplicial sphere is the boundary complex of a shellable simplicial ball [24].

There is a strong connection between shellings and certain stellar operations. Indeed, the theory of the so-called bistellar operations was essential for the proof of (1.4).

Definition 2. Let $\mathcal{M}$ be a simplicial n-manifold and let $\emptyset \neq A \in \mathcal{M}$ such that $\mathrm{link}(A; \mathcal{M}) = \mathcal{B}(B) \cdot \mathcal{L}$, where $B \neq \emptyset$ is a simplex not contained in $\mathcal{M}$. Then we call

$$\kappa_{(A,B)}\mathcal{M} := (\mathcal{M} \backslash A \cdot \mathcal{B}(B) \cdot \mathcal{L}) \cup \mathcal{B}(A) \cdot B \cdot \mathcal{L}$$

a stellar exchange.

Remarks, examples and additional notations. (1) Clearly $\kappa_{(A,B)}\mathcal{M}$ is again a simplicial n-manifold with $|\kappa_{(A,B)}\mathcal{M}| \cong |\mathcal{M}|$. Obviously $\kappa_{(A,B)}^{-1} = \kappa_{(B,A)}$ holds. The equivalence of simplicial manifolds by stellar exchanges is denoted by "$\approx_{\mathrm{stex}}$".

(2) If $\dim B = 0$, i.e. $B = \{b\}$ is a (new) point, then the operation $\kappa_{(A,B)} =: \sigma_{(A,b)} =: \sigma_A$ is known as *stellar subdivision* (see [10, 11, 15]). Here $A \in \mathrm{Bd}(\mathcal{M})$ or $A \in \mathrm{Int}(\mathcal{M})$ depending on whether $\mathcal{L}$ is a ball or a sphere. Conversely $\kappa_{(A,B)}^{-1} = \sigma_B^{-1}$ is an *inverse stellar subdivision* in the case $\dim A = 0$.

Clearly the definition of stellar subdivisions and their inverses is still applicable to arbitrary simplicial complexes. The stellar equivalence "$\approx_{\mathrm{st}\pm}$" is defined in the obvious way.

(3) $\kappa_{(A,B)} = \sigma_B^{-1}\sigma_A$ holds.

(4) If $\dim A + \dim B = n$ (i.e. $\mathcal{L} = \{\emptyset\}$) then $\kappa_{(A,B)} =: \chi_{(A,B)}$ is called a *bistellar k-operation* if $\dim A = k$. Obviously we have $\chi_{(A,B)}^{-1} = \chi_{(B,A)}$. The related equivalence relation is denoted by "$\approx_{\text{bst}}$".

If $\dim B \geqslant 1$, $B = p \cdot B'$, then $\chi_{(A,B)}$ is uniquely determined by p and the facet $F := A \cdot B'$ of $\mathcal{M}$. We then say that F is *visible* from p and we write $\chi(A, B) = \chi_{p/F}$. This expression comes from a special geometrical construction of bistellar equivalences for simplicial polytopes using a Bruggesser/Mani shelling process (see [5, 8]).

(5) $\mathcal{M} \xrightarrow{\text{sh,bst}} \mathcal{M}'$, $\mathcal{M} \xrightarrow{\text{sh}\pm\text{,bst}} \mathcal{M}'$ is defined in the obvious way. Note that these notations do not imply any order for the performance of the involved types of operations.

The concept of stellar subdivisions belongs to the standard tools in the theory of simplicial complexes. Later on we need the following fundamental theorem.

(1.5) For arbitrary simplicial complexes the following holds:

$$\mathcal{C}' \approx_{\text{st}\pm} \mathcal{C} \Leftrightarrow |\mathcal{C}'| \cong |\mathcal{C}| \quad \text{(Glaser (11])}$$

Obviously bistellar operations may be applied only in the interior of simplicial manifolds. As pointed out in [24] and as we shall see later on in this paper, they are in many respects the inner equivalents to shellings as boundary operations. Because of this and of (1.2), (1.3) the following result was rather surprising.

(1.6) The following holds for closed simplicial manifolds:

$$\mathcal{M}' \approx_{\text{bst}} \mathcal{M} \Leftrightarrow |\mathcal{M}'| \cong |\mathcal{M}| \quad [24]$$

For polytopes there are some stronger results.

(1.7) Let P be a simplicial d-polytope and let p be a vertex of P. Then there exists an equivalence

$$\chi_{p/F_r} \cdots \chi_{p/F_1} \mathcal{B}(P) = \mathcal{B}(T^d) \quad \text{(Ewald [8])}$$

For further information about bistellar operations and relationships to other problems the reader can consult [17, 22, 24, 25].

3. Shellings of balls

A first general construction method for simplicial balls which specializes the result (1.5) was given in [24]. We include it here for the sake of completeness.

Lemma 1. *For every simplicial n-ball $\mathcal{K}$ we have*

$$\mathcal{K} \xrightarrow{\text{sh}\pm\text{,bst}} \mathcal{F}(T^n)$$

Proof. From (1.6) follows $\mathcal{S} := \mathcal{K} \cup p \cdot \text{Bd}(\mathcal{K}) \approx_{\text{bst}} \mathcal{B}(T^{n+1})$, where p is a point joinable with $\mathcal{K}$. Then, following Theorem 5 in [21] there exists an equivalence $\chi_r \cdots \chi_1 \mathcal{S} = \mathcal{B}(P)$, where P is a simplicial (stacked) polytope and the χ_i-s are bistellar k-operations, $k \geq 1$. This implies that $p \in \chi_i \ldots \chi_1 \mathscr{S} =: \mathscr{S}_i$ for $i = 1, \ldots, r$. Write $\chi_i = \chi_{(A,B)}$ and consider the antistar of p. Then we observe the following. If $p \in B$ or $p \in A$ then the stellar operation is of the type $\chi_i = \chi_{p/F}$ or $\chi_i = \chi_{p/F}^{-1}$, respectively. This induces in the antistar of p the elementary shelling ρ_{+F} or ρ_{-F}, respectively. In the remaining case we have $A \in \operatorname{Int}(\operatorname{ast}(p; \mathscr{S}_{i-1}))$ and from this follows $\operatorname{ast}(p; \mathscr{S}_i) = \chi_i \operatorname{ast}(p; \mathscr{S}_{i-1})$. So we have $\mathscr{K} = \operatorname{ast}(p; \mathscr{S}) \xrightarrow{\text{sh}\pm,\text{bst}} \operatorname{ast}(p; \mathscr{B}(P))$ and our assertion now follows from the shellability of the antistars of vertices of polytopes. □

Remark. As already mentioned $\mathscr{K} \xrightarrow{\text{sh}} \mathscr{F}(T^n)$ does not hold in general ((1.2)). Obviously $\mathscr{F}(T^n) \xrightarrow{\text{sh}} \mathscr{K}$ can only hold for $\mathscr{K} = \mathscr{F}(T^n)$ and $\mathscr{K} \approx_{\text{bst}} \mathscr{F}(T^n)$ cannot hold if $\operatorname{Bd}(\mathscr{K}) \neq \mathscr{B}(T^n)$ because bistellar operations leave the boundary invariant. These examples show that in general one needs at least two types of the three operations "shellings, inverse shellings, bistellar operations" which are used in Lemma 1.

The following theorem improves Lemma 1.

Theorem 1. *Let $\mathscr{K}_1$, $\mathscr{K}_2$ be simplicial n-balls then*

$$\mathscr{K}_2 \approx_{\text{sh}\pm} \mathscr{K}_1.$$

Especially we have $\mathscr{K}_1 \approx_{\text{sh}\pm} \mathscr{F}(T^n)$.

Proof. Following Lemma 1 it is sufficient to prove our statement in the case $\mathscr{K}_2 = \chi_{(A,B)} \mathscr{K}_1$.

Because of (1.4) we can assume that there exists a shellable simplicial ball $\mathscr{K}$ with $\operatorname{Bd}(\mathscr{K}) = \operatorname{Bd}(\mathscr{K}_1) = \operatorname{Bd}(\mathscr{K}_2)$. Making stellar subdivisions in all those missing faces of $\operatorname{Bd}(\mathscr{K})$ which belong to $\operatorname{Int}(\mathscr{K})$ we get a simplicial ball $\mathscr{K}'$ with the following properties.

(a) $\mathscr{K}'$ is shellable (Lemma 5 in [24])
(b) $\operatorname{Bd}(\mathscr{K}') = \operatorname{Bd}(\mathscr{K})$
(c) $\operatorname{Bd}(\mathscr{K}')$ *is full in* $\mathscr{K}'$ (see [14]), i.e. every simplex of $\mathscr{K}'$ with all its vertices contained in $\operatorname{Bd}(\mathscr{K}')$ is itself contained in $\operatorname{Bd}(\mathscr{K}')$

From (b), (c) follows that $\mathscr{S}_i := \mathscr{K}_i \cup \mathscr{K}'$ is a simplicial n-sphere for $i = 1, 2$.

Because of (a) we can write $\mathscr{K}' = \rho_{+F_r} \ldots \rho_{+F_1} \mathscr{F}(F_0)$, F_0 an n-simplex, from which follows $\mathscr{S}_1 \setminus \{F_0\} = \rho_{+F_1} \ldots \rho_{+F_r} \mathscr{K}_1$ (compare Lemma 5 in the next section). Choose a facet F_A in $\operatorname{st}(A; \mathscr{S}_1) = \operatorname{st}(A; \mathscr{K}_1)$. Then there exists a sequence $F_0 = E_0, E_1, \ldots, E_s = F_A$ of facets of $\mathscr{S}_1$ such that $E_{i-1} \cap E_i$ is a common facet of E_{i-1}, E_i for $i = 1, \ldots, s$ (this well-known property of simplicial spheres is called "strongly connected"). With this sequence we get

$$\mathscr{S}_1 \setminus \{F_A\} = \rho_{+E_{s-1}} \rho_{-E_s} \ldots \rho_{+E_{i+1}} \rho_{-E_i} \ldots \rho_{+E_0} \rho_{-E_1} (\mathscr{S}_1 \setminus \{F_0\})$$

Obviously the following additional transformation

$$\mathscr{S}_1 \setminus \{F_A\} \xrightarrow{\text{sh}} \mathscr{S}_1 \setminus \text{st}(A; \mathscr{S}_1)$$

holds (this is a special case of Lemma 5 in the next section). So we have proved $\mathscr{K}_1 \approx_{\text{sh}\pm} \mathscr{S}_1 \setminus \text{st}(A; \mathscr{S}_1)$.

Analogously we get $\mathscr{K}_2 \approx_{\text{sh}\pm} \mathscr{S}_2 \setminus \text{st}(B; \mathscr{S}_2) = \mathscr{S}_1 \setminus \text{st}(A; \mathscr{S}_1)$ which completes the proof. □

As an easy consequence we get the following improvement of (1.6) for simplicial spheres (compare (1.7)).

Corollary 1. *Let $\mathscr{S}$ be a simplicial n-sphere and $p \in \text{vert}(\mathscr{S})$. Then there exists a transformation*

$$\chi^{\pm}_{p/F_r} \cdots \chi^{\pm}_{p/F_1} \mathscr{S} = \mathscr{B}(T^{n+1}).$$

Clearly the methods used above cannot be applied to general manifolds with boundary. But we shall prove similar results for manifolds with the help of other methods developed in the next section.

4. Transformations of manifolds with boundary

At the beginning of this section we shall enumerate some basic construction methods which may be of intrinsic interest.

Lemma 2. *Let $\mathscr{M}$ be a simplicial n-manifold and $\mathscr{K} \subseteq \mathscr{M}$ a shellable n-ball. Then the following holds*

$$\mathscr{M} \approx_{\text{bst}} (\mathscr{M} \setminus \text{Int}(\mathscr{K})) \cup p \cdot \text{Bd}(\mathscr{K})$$

Proof. By our assumption follows the existence of an inverse shelling $\mathscr{K} = \rho_{+A_m} \cdots \rho_{+A_1} \mathscr{F}(F_0)$.

From this we obtain by induction on m:

$$(\mathscr{M} \setminus \text{Int}(\mathscr{K})) \cup p \cdot \text{Bd}(\mathscr{K}) = \chi_{A_m} \cdots \chi_{A_1} \chi_{(F_0, p)} \mathscr{M}. \quad \square$$

Lemma 3. *Let $\mathscr{M}$ be a simplicial n-manifold, $A \in \text{Int}(\mathscr{M})$ and $p \in \text{link}(A; \mathscr{M})$ such that*

(1) $\text{ast}(p; \text{link}(A; \mathscr{M}))$ *is shellable*

(2) $\text{link}(p; \mathscr{M}) \cap \text{Int}(\text{ast}(p; \text{link}(A; \mathscr{M}))) = \{\emptyset\}$

Then we have

$$\mathscr{M} \approx_{\text{bst}} (\mathscr{M} \setminus \text{st}(A; \mathscr{M})) \cup p \cdot \text{ast}(p; \text{link}(A; \mathscr{M}) \cdot \mathscr{B}(A))$$

Proof. See Lemma 1 in [21], Lemma 1 in [24]. □

Lemma 4. *Let $\mathcal{M}$ be a simplicial n-manifold and let $\mathcal{K} \subseteq \mathrm{Bd}(\mathcal{M})$ be a shellable $(n-1)$-ball. Then we have*

$$\mathcal{M} \cup p \cdot \mathcal{K} \xrightarrow{\text{sh}} \mathcal{M}.$$

Proof. Given a shelling $\rho_{-F_1} \ldots \rho_{-F_m}\mathcal{K} = \mathcal{F}(F_0)$ of $\mathcal{K}$, we get $\mathcal{M} = \rho_{-p \cdot F_0}\rho_{-p \cdot F_1} \ldots \rho_{-p \cdot F_m}(\mathcal{M} \cup p \cdot \mathcal{K})$ □

Lemma 5. *Let M be a simplicial n-manifold, $\mathcal{K} \subseteq \mathrm{Bd}(\mathcal{M})$ a shellable $(n-1)$-ball and $\mathcal{K} \subseteq \mathrm{link}(p; \mathcal{M})$ for a vertex $p \in \mathrm{Int}(\mathcal{M})$. Then*

$$\mathcal{M} \xrightarrow{\text{sh}} \mathcal{M} \backslash p \cdot \mathrm{Int}(\mathcal{K}) =: \mathcal{M}'$$

Proof. Let $\rho_{-F_1} \ldots \rho_{-F_m}\mathcal{K} = \mathcal{F}(F_1)$ be a shelling of $\mathcal{K}$. By induction on m one obtains:

$$\rho_{-p \cdot F} \ldots \rho_{-p \cdot F_m}\mathcal{M} = \mathcal{M} \backslash p \cdot \mathrm{Int}(\mathcal{M}) =: \mathcal{M}' \quad \text{and}$$
$$\mathrm{Bd}(\mathcal{M}') = (\mathrm{Bd}(\mathcal{M}) \backslash \mathrm{Int}(\mathcal{K})) \cup p \cdot \mathrm{Bd}(\mathcal{K}). \quad \square$$

Now we are able to replace, under certain niceness conditions, stellar subdivisions by elementary operations.

Lemma 6. *Let $\mathcal{M}$ be a simplicial n-manifold and $A \in \mathrm{Int}(\mathcal{M})$. If* link$(A; \mathcal{M})$ *is shellable then*

$$\mathcal{M} \approx_{\text{bst}} \sigma_A \mathcal{M}.$$

Proof. this follows immediately from Lemma 3 (see Theorem 1 in [21], Lemma 2 in [24]). □

Lemma 7. *Let $\mathcal{M}$ be a simplicial n-manifold and $A \in \mathrm{Bd}(\mathcal{M})$. If both* link$(A; \mathcal{M})$ *and* link$(A; \mathrm{Bd}(\mathcal{M}))$ *are shellable then*

$$\sigma_A \mathcal{M} \xrightarrow{\text{sh,bst}} \mathcal{M}.$$

Proof. Following Lemma 4 the shellability of clst$(A; \mathrm{Bd}(\mathcal{M}))$ implies $\mathcal{M}' := \mathcal{M} \cup p \cdot \mathrm{clst}(A; \mathrm{Bd}(\mathcal{M})) \xrightarrow{\text{sh}} \mathcal{M}$. Furthermore the shellability of link$(A; \mathcal{M})$ implies the shellability of ast$(p; \mathrm{link}(A; \mathcal{M}')) = \mathcal{B}(A) \cdot \mathrm{link}(A; \mathcal{M})$ and it is easy to see that (2) of Lemma 3 holds too. So we get

$$\mathcal{M}' \approx_{\text{bst}} (\mathcal{M}' \backslash \mathrm{st}(A; \mathcal{M}')) \cup p \cdot \mathcal{B}(A) \cdot \mathrm{link}(A; \mathcal{M})$$
$$= (\mathcal{M} \backslash \mathrm{st}(A; \mathcal{M})) \cup p \cdot \mathcal{B}(A) \cdot \mathrm{link}(A; \mathcal{M}) \approx \sigma_A \mathcal{M} \quad \square$$

Lemma 8. *Let $\mathcal{M}$ be a simplicial n-manifold and $A \in \mathrm{Bd}(\mathcal{M})$. If both* link$(A; \mathcal{M})$ *and* link$(A; \mathrm{Bd}(\mathcal{M}))$ *are shellable then*

$$\mathcal{M} \xrightarrow{\text{sh,bst}} \sigma_A \mathcal{M}.$$

Proof. Following Lemma 3 the shellability of $\mathrm{clst}(A;\mathcal{M})$ implies $\mathcal{M} \approx_{\mathrm{bst}} (\mathcal{M}\setminus \mathrm{st}(A;\mathcal{M})) \cup p \cdot \mathrm{Bd}(\mathrm{clst}(A;\mathcal{M})) =: \mathcal{M}'$. Now $\mathrm{clst}(A;\mathrm{Bd}(\mathcal{M}')) = \mathrm{clst}(A;\mathrm{Bd}(\mathcal{M}))$ is a shellable n-ball which is contained in $\mathrm{link}(p;\mathcal{M})$. So we obtain by Lemma 5

$\mathcal{M}' \xrightarrow{\mathrm{sh}} \mathcal{M}'\setminus p \cdot \mathrm{st}(A;\mathrm{Bd}(\mathcal{M}')) \cong \sigma_A\mathcal{M}$. □

Using the shellability of simplicial 2-spheres we get as an easy consequence:

Lemma 9. *Let $\mathcal{M}$, $\mathcal{M}'$ be simplicial n-manifolds, $n \leq 4$. Then*

$$|\mathcal{M}'| \cong |\mathcal{M}| \Leftrightarrow \mathcal{M}' \xrightarrow{\mathrm{sh,bst}} \mathcal{M}.$$

Especially we have $\mathcal{F}(T^n) \xrightarrow{\mathrm{sh,bst}} \mathcal{K}$ and $\mathcal{K} \xrightarrow{\mathrm{sh,bst}} \mathcal{F}(T^n)$ for every simplicial n-ball $\mathcal{K}$, $n \leq 4$.

Proof. This follows directly from (1.5) and Lemmas 6, 7, 8 with the help of (1.1) and the fact that every 2-sphere is polytopal and hence shellable (see Steinitz Theorem [12]). □

Before we shall prove this result in arbitrary dimensions we formulate two more lemmas which are needed.

Lemma 10. *Let $\mathcal{M}$ be a simplicial n-manifold and*

$$\kappa_{(A,B)}\mathcal{M} = (\mathcal{M}\setminus A \cdot \mathcal{B}(B) \cdot \mathcal{L}) \cup \mathcal{B}(A) \cdot B \cdot \mathcal{L} \quad \textit{and}$$
$$\kappa_{(C,D)}\mathcal{L} = (\mathcal{L}\setminus C \cdot \mathcal{B}(D) \cdot \mathcal{L}') \cup \mathcal{B}(C) \cdot D \cdot \mathcal{L}'.$$

Then the following holds

(1) $\kappa_{(B\cdot C,D)}\kappa_{(A,B)}\mathcal{M} = \kappa_{(A,B)}\kappa_{(A\cdot C,D)}\mathcal{M}$
(2) $\mathrm{link}(B \cdot C; \kappa_{(A,B)}\mathcal{M}) = \mathcal{B}(A) \cdot \mathcal{B}(D) \cdot \mathcal{L}'$
(3) $\mathrm{link}(A \cdot C; \mathcal{M}) = \mathcal{B}(B) \cdot \mathcal{B}(D) \cdot \mathcal{L}'$
(4) $\mathrm{link}(A; \kappa_{(A\cdot C,D)}\mathcal{M}) = \mathcal{B}(B) \cdot \kappa_{(C,D)}\mathcal{L}$

This is an easy exercise and only a slight generalization of Lemma 3 in [24].

Lemma 11. *Let $\mathcal{C}$ be a simplicial complex. Then there exists a unique decomposition $\mathcal{C} = \mathcal{B}(P) \cdot \mathcal{C}'$ such that P is a simplexoid (i.e. $\mathcal{B}(P) = \mathcal{B}(T_1) \cdot \ldots \cdot \mathcal{B}(T_r)$, T_k – s simplices) and P is maximal with this property.*

Proof. See in [23]. □

Remark. Clearly, if P_1, P_2 are simplexoids then $\mathcal{B}(P_1) \cdot \mathcal{B}(P_2)$ is again isomorphic to the boundary complex of a simplexoid.

Theorem 2. *Lemma 9 holds in arbitrary dimensions.*

Proof. The suffiency follows at once from Remark (2) for Definition 1 and

Remarks (1), (4) for Definition 2. In order to prove the existence of our transformation we can assume $\mathcal{M}' = \kappa_{(A,B)}\mathcal{M}$ (apply (1.5) and Remark (2) for Definition 2).

Now let $\operatorname{link}(A;\mathcal{M}) = \mathcal{B}(B)\cdot\mathcal{L}$ and let $\mathcal{L} = \mathcal{B}(P)\cdot\mathcal{L}'$ be the unique decomposition of $\mathcal{L}$, according to Lemma 11. If $m := \dim\mathcal{L}' \leq 2$ then $\mathcal{L}'$ is shellable which implies the shellability of $\mathcal{L}$. From this and $\kappa_{(A,B)} = \sigma_B^{-1}\sigma_A$ we conclude our assertion immediately with the help of Lemmas 6, 7 and 8. Otherwise proceed by induction on m. Using (1.5) we get an equivalence $\kappa_r \ldots \kappa_1\mathcal{L}' = \mathcal{B}(T^{m+1})$ or $\mathcal{F}(T^m)$. We continue the proof with induction on r.

If $r \leq 1$ then $\mathcal{L}'$ is shellable and we can use the same arguments as above. Otherwise let $\kappa_1 = \chi_{(C,D)}$. Then we may apply κ_1 to $\mathcal{B}(P)\cdot\mathcal{L}'$ and get $\kappa_{(C,D)}(\mathcal{B}(P)\cdot\mathcal{L}') = \mathcal{B}(P)\cdot\kappa_1\mathcal{L}'$. Two cases arise.

Case 1. $D \notin \mathcal{M}$.

Following Lemma 10 we get:

(1) $\kappa_{(A,B)}\kappa_{(A\cdot C,D)}\mathcal{M} = \kappa_{(B\cdot C,D)}\kappa_{(A,B)}\mathcal{M}$

(2) $\operatorname{link}(B\cdot C; \kappa_{(A,B)}\mathcal{M} = \mathcal{B}(A)\cdot\mathcal{B}(D)\cdot\mathcal{B}(P)\cdot\mathcal{L}''$ where $\operatorname{link}(C;\mathcal{M}') = \mathcal{B}(D)\cdot\mathcal{L}''$. As $\dim\mathcal{L}'' < m$ we get by induction $\kappa_{(B\cdot C,D)}\kappa_{(A,B)}\mathcal{M} \xrightarrow{\text{sh,bst}} \kappa_{(A,B)}\mathcal{M}$.

(3) $\operatorname{link}(A\cdot C;\mathcal{M}) = \mathcal{B}(B)\cdot\mathcal{B}(D)\cdot\mathcal{B}(P)\cdot\mathcal{L}''$. Consequently we get again by induction on m

$$\mathcal{M} \xrightarrow{\text{sh,bst}} \kappa_{(A\cdot C,D)}\mathcal{M}$$

(4) $\operatorname{link}(A; \kappa_{(A\cdot C,D)}\mathcal{M}) = \mathcal{B}(B)\cdot\mathcal{B}(P)\cdot\kappa_1\mathcal{L}'$. Here we obtain

$$\kappa_{(A\cdot C,D)}\mathcal{M} \xrightarrow{\text{sh,bst}} \kappa_{(A,B)}\kappa_{(A\cdot C,D)}\mathcal{M}$$

by the inductive assumption concerning r, if there is no proper decomposition of $\kappa_1\mathcal{L}'$ in the sense of Lemma 11, and by the inductive assumption concerning m otherwise (see remark for Lemma 11!).

From (1)–(4) clearly follows $\mathcal{M} \xrightarrow{\text{sh,bst}} \chi_{(A;B)}\mathcal{M}$. We remark that $D \notin \mathcal{M}$ if $\dim D = 0$.

Case 2. $D \in \mathcal{M}$. This case can be reduced to Case 1. As mentioned above we may assume $\dim D \geq 1$. Let $D = p\cdot E$, p a vertex of D. Then we subdivide $\mathcal{L}'$ in the 0-face p (which clearly yields an isomorphic complex), $\kappa_{(p,q)}\mathcal{L}' = (\mathcal{L}' \setminus p\cdot\operatorname{link}(p;\mathcal{L}')) \cup q\cdot\operatorname{link}(p;\mathcal{L}')$, where q is a new vertex not contained in $\mathcal{M}$. Then we derive from Lemma 10:

(1) $\kappa_{(B\cdot p,q)}\kappa_{(A,B)}\mathcal{M} = \kappa_{(A,B)}\kappa_{(A\cdot p,q)}\mathcal{M}$

(2, 3) Analogously as in Case 1 we get by the inductive assumption

$$\kappa_{(B\cdot p,q)}\kappa_{(A,B)}\mathcal{M} \xrightarrow{\text{sh,bst}} \kappa_{(A,B)}\mathcal{M} \quad\text{and}\quad \mathcal{M} \xrightarrow{\text{sh,bst}} \kappa_{(A\cdot p,q)}\mathcal{M}.$$

(4) $\operatorname{link}(A; \kappa_{(A\cdot p,q)}\mathcal{M}) = \mathcal{B}(P)\cdot\mathcal{B}(D)\cdot\kappa_{(p,q)}\mathcal{L}'$. As $\kappa_{(p,q)}\mathcal{L}' \approx \mathcal{L}'$ we may transform $\kappa_{(p,q)}\mathcal{L}'$ into $\mathcal{B}(T^{m+1})$ or $\mathcal{F}(T^m)$, respectively, with stellar exchanges κ_i'

which correspond to the κ_i, $i = 1, \ldots, r$. Here we have $\kappa_1' = \kappa_{(C,q\cdot E)}$.

Now the situation has changed as $q\cdot \notin \kappa_{(A\cdot p,q)}\mathcal{M}$. Thus we can apply Case 1 to get

$$\kappa_{(A\cdot p,q)}\mathcal{M} \xrightarrow{\text{sh,bst}} \kappa_{(A,B)}\kappa_{(A\cdot p,q)}\mathcal{M}.$$

This completes the proof. □

Corollary 3. *Let $\mathcal{K}$ be a simplicial n-ball. Then we have $\mathcal{K} \xrightarrow{\text{sh,bst}} \mathcal{F}(T^n)$ as well as $\mathcal{F}(T^n) \xrightarrow{\text{sh,bst}} \mathcal{K}$.*

We strongly believe that Theorem 1 holds too for manifolds with boundary, but we have not made great effort to prove this. In the proof of Theorem 1 we can clearly use now Theorem 2 instead of Lemma 1. This leads directly to the following generalization.

Theorem 3. *Let $\mathcal{M}$, $\mathcal{M}'$ be simplicial manifolds whose boundaries are spheres, then*

$$|\mathcal{M}| \cong |\mathcal{M}'| \Leftrightarrow \mathcal{M} \approx_{\text{sh}\pm} \mathcal{M}'$$

Note added in proof

Theorem 3 holds for manifolds with arbitrary nonempty boundary. The proof which is quite different from that of Theorem 3 will be published in the near future.

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